Concrete Representations and Strategies for Solving Linear Equations

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This paper is a report of students' responses to instruction with concrete representations in solution to linear equations. The sample consisted of 21 Year 8 students from a middle-class suburban state secondary school with a reputation for high academic standards and innovative mathematics teaching. The students were interviewed before and after instruction. Interviews and classroom interactions were observed and videotaped. A qualitative analysis of the responses revealed that most students did not use the materials in solving problems. The increased processing load caused by concrete representations is hypothesised as the reason.

Many theories of mathematics education are based on the use of concrete representations as manipulatives for making connections with mathematical ideas. Hiebert's (1988) theory stressed the importance of making connections "between the written marks on paper and the quantities or actions they represent" (p. 336) and preserving the "relevant properties when connecting referent to symbol" (p. 338). Leinhardt's (1988) model involved connections between four knowledge types, one of which was concrete. Bloomer and Carlson's (1993) theory stated that the abstract stage followed understanding of the concrete and connecting stages.

Boulton-Lewis (1993) acknowledged the usefulness of concrete analogues (representations) in reducing learning effort, mediating transfer between tasks and situations, and indirectly facilitating transition to higher levels of abstraction, however she suggested that if the analogs were not well known they could be a disadvantage. She observed that concrete representations often failed to produce expected results, arguing that this was due to the processing load required to map them into a mental model. This finding was supported by Hart's (1989) subtraction research where she found that there appeared to be a gap between the use of concrete representations and symbolic mathematical language. She stressed that the methods employed when using materials should be translatable into the algorithm.

The proposals with regard to the use of concrete representations in algebra (e.g, Chalouh & Herscovics, 1988; Quinlan, Low, Sawyer & White, 1993; Thompson, 1988) take into account the relationship between algebra and number. They recommend representing numbers with counters and variables with empty containers such as cups. However, because algebra knowledge quickly becomes abstract, manipulative techniques can become artificially complex (e.g., the representation of negative numbers by Thompson, 1988). Halford and Boulton-Lewis (1992) argued that recognition of correspondences between concrete representations and mathematical symbols and between number and algebra depended on a series of multiple system mappings for which processing loads are high.

Hence, questions remain: are concrete representations effective in teaching algebra; if so, which ones and why; what is the effect of processing load in such use?

In an attempt to provide answers, a research project (Atweh, Boulton-Lewis, & Cooper, 1994) was undertaken to investigate how instruction with concrete representations affects understanding of variables, expressions and equations, the equals sign, and solution of linear equations. This paper describes, in detail, students' understanding of and strategies for solving linear equations and relates these to their understanding of the use of concrete representations.

Method

Sample

This consisted of a class of 21 students (7 girls and 14 boys) from a year 8 class of mixed abilities in a middle-class state secondary school with a reputation for high academic standards and innovative mathematics teaching. The class had completed an algebra unit which used patterns to introduce variable as generalisation and cups and counters to introduce variable as unknown.

Procedure

The students were interviewed individually before instruction (the pre-interview). The interviews took about 30 minutes and were videotaped. The lessons were observed and videotaped. One month after instruction, each student was re-interviewed (the post-interview).

The interviews consisted of tasks for: patterning and generalisation, meaning of variable, differences between expression and equation, meaning of equals, solution of linear equations, inverse operations, and the use of concrete and pictorial representations. For this paper only responses for solving the linear equation $2x + 5 = 17$ are described and discussed. The students were asked to solve the equation and if they did not voluntarily use the concrete representations they were then asked to do so. Students were provided with the materials (cups, counters, and sticks) used by the teacher in instruction on linear equations as well as pen and paper. After solving the equation in the post-interview students were questioned with regard to perceived usefulness of these materials.

Instruction

Instruction was based on concrete representations consisting of green discs to represents units, yellow discs to represent negative units, white cups to represent variables and yellow cups to represent the negative value of variables (Thompson, 1988) and diagrams. There was some inconsistency on the part of the teacher in using either the cup or the unknown number of discs inside the cup to represent the variable. The notion of upsetting and restoring balance and sharing discs equally between cups to solve equations was introduced. Using a diagram on the board the teacher depicted the concrete represention of the equation and changes to the equation using a line to separate each side. Throughout the lessons the materials were available at the front of the room, however very few students asked to use the materials to solve classroom examples preferring to work mentally or use the diagram procedures.

In the first lesson the teacher gave examples where written statements such as "I had a bag of marbles and I lost half of them" were represented mathematically as "/2". In the second and third lessons, the teacher set the aim as modelling mathematical expressions and gave formal rules to guide the representations. In lesson 4, the teacher showed how to solve the equation $x + 5 = 7$ using the balance perspective. In lesson 5, the teacher introduced what he called a short cut. Rather than changing subtraction to adding a negative, he showed that this was equivalent to doing the reverse operation of both sides (e.g., $3x - 5 = 7$ was changed by adding 5 to both sides).

Results

Analysis

All interviews were analysed using the computer program Non-numerical Unstructured Data Indexing Searching and Theory-building (NUD.IST) version 3.04. The analysis accepted as correct manipulation of material where sticks or differently coloured counters replaced cups.

The categories for each task are described below. Interview extracts that illustrate, supplement, and add greater meaning to each category are found in Appendix A.

Equation representation

Nineteen students could not represent the equation 2x + 5 = 17 correctly with materials during the pre-interview. Nine students had no idea how to use the materials, six showed an incomplete representation where only one side of the equation was represented, and four displayed the equation literally, i.e. representing 2x as 2 counters and one cup or one object. Only two students showed a correct representation of the equation and they used mixed objects.

In the post-interviews, students used similar methods. Although 17 students could still not represent the equation correctly, eight showed an incomplete representation, three represented the equation literally, and six could not represent it at all. For the correct representations, cups were used once and mixed objects were used three times. (See Appendix A)

A comparison of pre to post-interviews showed that three students had moved from an incorrect representation to a correct representation. Two of these students initially represented the equation literally but used either cups or mixed objects correctly for the post-interview; the other student went from no idea to correct use of mixed objects. Only one student was correct for both interviews, using mixed objects each time. It is interesting to note that one student who could represent the equation correctly with mixed objects for the pre-interview, showed an incomplete representation for the post-interview. Sixteen students either could not or would not represent the equation on either occasion.

Initial Strategies Used In Solving The Equation

Students were presented with the equation 2x + 5 = 17 and asked to solve it using any means they wished. Analysis of pre-testing solution methods revealed 14 students could solve the equation correctly with 13 of these employing mental strategies. Of these ten chose to use an inverse mental strategy which involved reversing the operations in the equation, i.e. subtracting 5 from 17 to get 12 and dividing this number by 2 to get 6 for their answer. Another three incorporated a heuristic (trial and error) mental approach by substituting several numbers for x until the correct one to fit the equation was reached. Only one student used materials to arrive at a correct answer. However, this student did not use a balance approach. Instead, he counted up from 5 using counters, found that 12 were needed to reach 17, then divided the 12 into two groups to get 6 for their answer. Seven students could not solve the equation correctly during the pre-interview. Six of these students showed a lack of knowledge that 2x is a multiplication operation, as they added the 2 and 5 in the left side of the equation and took this from 17 to get 10, which they said was the value of x. Another student used a combination of materials and a mental calculation to reach 5 as the value of x. This student had represented the left side of the equation literally but did not represent the 17. He added mentally the 2 and 5 to get 7 and said that x must equal 5.

By the post-interview, students used only two correct solution strategies for initial equation solution. These were the inverse mental strategy (14) and the heuristic approach (1) described above. Five students used the incorrect mental approach of adding the 2 and 5 and taking this from 17 to get 10 for x and one student wrote this incorrect mental approach on paper. (See Appendix A).

A comparison of pre and post-interview strategies showed that 13 students were correct on both occasions. Twelve of these used the inverse mental strategy and one moved from using materials and counting in the pre-interview to the inverse mental strategy in the post-interview. Only two students showed an improvement and they moved from using the incorrect mental strategy, i.e. adding 2 and 5 to get 7 and taking this from 17 to get 10 for x, during the pre-interview to using the correct inverse mental strategy for the post-interview. Five students were incorrect on both occasions and one student who had employed the heuristic approach for the pre-interview actually employed the incorrect mental strategy for the post-interview.

Material Use for Solving the Equation

When students were asked to solve the equation with concrete representations, two types of material use were identified from the pre and post-interviews. The first strategy was generative where students employed materials, using the balance perspective, to arrive at an answer. Only one student generated an answer during the pre-interviews. This student represented the equation literally, but ignored the object used to represent x and used the other objects to generate her answer. For the post-interview, five students generated answers. One was the generative student from the pre-interview who continued to use mixed objects to represent the equation and generate the answer. Of the other four students, one represented the equation literally but was still able to generate an answer, two others used mixed objects to represent the equation and generate an answer and one student correctly used cups to represent the equation and generate the answer. The second strategy employed to solve the equation was illustrative where the operations used in the mental strategies were applied, sometimes in an incomplete way. Some students started with two groups of 6 discs, added 5 and got 17. Other students used the inverse strategy of starting with 17 discs, subtracting 5 and dividing by 2 (without using cups). Six students illustrated their answers during the pre-interview and ten students during the post-interview. One student was an exception during the post-interviews. She used mixed objects to represent the equation but went on to illustrate the answer. Fourteen students during the pre-interviews and seven students in the post-interviews could not or would not use the materials to solve the equation. (See Appendix A)

A comparison of pre- and post-interview responses showed only one student was able to use the materials correctly to generate an answer on both occasions, two students went from illustrating to generating, and another initially had no idea how to use the materials but generated an answer during the post-interview. The majority of students (7), however, could not use the materials to solve the equation for either the pre- or post-interviews. Six students moved from having no idea how to use the materials during the pre-interviews to illustrating an answer for the post-interviews, while another four students illustrated to find their answer on both occasions.

Students' perceptions of material use

After solving the equation during the post-interview, students were asked if they thought using materials had been helpful. Three students responded favourably to use of materials. Two of these students simply replied "Yes" while the other student said, "Instead of having to use your head, you can use either the counters to keep the number on the desk and then you can , after you've used them you just write it down on your paper."

Eight students felt materials were no use at all. This was evidenced through responses such as, "I'm no good at it" (using the materials), "No, I can do them in my head", and "No. Well, I find it a bit easier to sort of work it out on paper, because sometimes the things [materials] confuse you a bit. I find it easier to just write it straight down on paper". Using materials some of the time was an option that ten students preferred. Most of these responses evidenced prior working out, either mentally or with pen and paper, before using materials. For example, "Now I can do them in my head and I work it out" and "When I can't do them in my head [I use materials]", while another student answered simply, "I usually just do it on paper".

Discussion and conclusions

The results in this study are unequivocal at one level - the students did not use the knowledge taught to them about the concrete representations. Only one of the twenty-one students correctly used cups and counters to represent the equation. At the post-interview, no students voluntarily used materials. When directly asked to use them, only four of the twenty-one students could use the materials to generate an answer.

Why was this? When given the choice students preferred the mental approach (predominantly inverse) which met their needs simply and effectively. As a result most uses of concrete representations were illustrative because mental strategies overrode any knowledge of the generative use of materials.

The reason for this appears to lie within the processing loads associated with concrete representations (Boulton-Lewis, 1993) and transferring understandings from arithmetic to algebra (Halford & Boulton-Lewis, 1992). Even during instruction, very few students used materials. This is not surprising if the material solution to 2x - 3 = 5 is analysed. First, the student has to recognise these symbols as representing an equation, with one side equivalent to the other, and including two operations, the x2 and the -3. Second, the student has to replace this arithmetic representation with a concrete representation, putting out two cups (recognising '2x' as two x's) and three yellow counters (recognising that '- 3' is '+ -3') on the left and 5 green counters on the right. Third, the student has to identify the 'addition of -3' as the best avenue of attack to simplify the equation. Fourth, the student has to represent this concretely by adding three green counters to both sides (to remove the -3). Fifth, the student has to recognise that dividing by two will simplify the equation to the desired point. Sixth, the student has to understand that division by 2 is partitioning the eight green counters into two sets of four counters and undertake this.

The difficulty for students is that they have to integrate knowledge of laws and relations of arithmetic, knowledge of the mathematical meaning of equals and equation, knowledge of variable, and knowledge of methods to concretely represent variables, numbers and operations by cups and counters. It is the argument of this paper that this use of materials imposes a significantly greater processing load than mentally visualising 2x - 3 = 5 as multiplying by 2 and subtracting 3 and then reversing these operations (i.e., adding 3 and dividing by 2) to find the solution. Students' negative perceptions of the usefulness of concrete representations further highlight the processing load incurred when using materials to solve equations.

It would appear evident that the processing load of the material usage procedure for solving linear equations would be lower (and not as confusing) if it did not involve materials for negative variables and negative numbers. It also appears evident that if use of materials were restricted to extending operations to variables (e.g., representing multiplication of a variable, 2x, by two cups and addition of variable and number, x+3, by a cup and three counters) then processing load would not be as high as for the solution of linear equations. Hence, the results of this paper do not necessarily mean that materials have no place in teaching algebra. They do tend, however, to point out the dangers of unthinking dependence on materials in teaching. Further research is needed to delineate the effective use of concrete representations, and the role of pictorial and mental representations, in learning and teaching algebra.

References

Atweh, W., Boulton-Lewis, G.M., & Cooper, T.J. (1984, July). Representations, strategies, cognitive load and effective learning and teaching of linear equation solution procedures. In the Proceedings of the Seventeeth Annual Conference of the Mathematics Education Research Group in Australasia, *Lismore, NSW.*

Bloomer, A. & Carlson, P. (1993). Activity math: Using manipulatives in the classroom. *Menlo Park, CA: Addison-Wesley.*

Boulton-Lewis, G.M. (1993). Young children's representations and strategies for subtraction. British Journal of Educational Psychology, *64, 451-456.*

Chalou, L. & Herscovics, N. (1988). Teaching algebraic expressions in a meaningful way. In A. Coxford & A.P. Shulte (Eds.), The ideas of algebra, K-12 *(1988 Yearbook: 33-42). Reston, VA: National Council of Teachers of Mathematics.*

Hart, K. (1989). There is little connection. In P. Ernest (Ed.), Mathematics teaching: The state of the art *(138-142). New York: Falmer Press.*

Halford, G.S. & Boulton-Lewis, G.M. (1992). Value and limitations of analogs in teaching mathematics. In A. Demetriou, A. Efklides and M. Shayer (Eds.), Modern theories of cognitive development go to school. *London: Routledge.*

Hiebert, J. (1988). A theory of developing competence with written mathematical symbols. Educational Studies in Mathematics, *19, 333-355.*

Leinhardt, G. (1988). Getting to know: Tracing students' mathematical knowledge from intuition to competence. Educational psychologist, *23 (2), 119-144.*

Quinlan, C., Low, B., Sawyer, T., & White, P. (1993). A concrete approach to algebra. *Sydney, Australia: Mathematical Association of New South Wales.*

Thompson, F.M. (1988). Algebraic instruction for the younger child. In A. Coxford & A.P. Shulte (Eds.), The ideas of algebra, K-12 *(1988 Yearbook: 33-42). Reston, VA: National Council of Teachers of Mathematics.*

Appendix A

Equation representation

Correct Representations: Cups - the student placed two empty cups on the table and said "x is equal to one white cup and then 2x there, and then you add 5 there (added 5 yellow discs), and then that's equal to 17 (added 17 yellow discs)".

Mixed objects - the student made a group of 17 sticks and said "That's 17 stripes, then I have to have 5 (made a group of five blue discs), and then 2x, they resemble ... that's 1x, that's 2x (added two green discs)".

Incorrect Representations: Incomplete - the student placed two cups on the table, put discs in each, placed five green discs near the two cups on the table (but didn't put out 17 discs to complete the equation), stated "Okay, equals 17 and there's 12 in both of these cups" and replied to a question about the 17 by saying "There should be six in each cup, so the 12 plus the 5 is the 17".

Literal - the student placed 2 sticks on the table, said "So there's 2x and x has to be different", placed a pile of green discs next to the sticks, and stated "So there's two (points to the 2 sticks) and there's the x (points to the green discs), and plus 5 (placed 5 sticks beside the group of discs), equals 17 which should be in the red as well".

Initial strategies used in solving the equation

Correct Strategies: Inverse Mental - the student said "I sort of took 5 away from 17, which was 12, and then 12 divided by 2, which equals 6".

Heuristic - the student said "Six!", the interviewer asked, "How did you work that out?", the student said "Well, I started off with - I went 2 times 1 equals 2 plus 5. It doesn't equal so many. Then I did that till I got up to 6. I went 2 times 6 plus 5 equals 17", the interviewer asked "Did you try all the ones in between? Did you try 3, 4, 5, and 6?" and the student replied "Yes".

Material, Counting Up - the student stated "Six. I got the 5 (5 blue discs), then I figured out - I counted up to 17 with discs, but then I counted how many counters there were and divided it by 2".

Incorrect Strategies: Mental Calculation - the student said "Ten", the interviewer asked "How did you get ten?" and the student stated "Because 2 and 5 is 7, and 'x' is 10, so it adds up to 17".

Material and Mental - the student said "There'd be 2 (placed 2 yellow discs on the desk), then x, then the plus sign, then the 5 (5 discs), and then you'd get, like the 7, those two equal the 7. Then you've got your answer on the other side. 'x' would be 5. I plussed 2 times 5 was 10, and then you add the 2 and the 5, and that gives you the 17".

Paper - the student wrote $2x + 5 = 17$, said "Well I've got the 2x plus 5 and then the - you've got the 5, that equals 17, and then you'll have to find the x, which is, the 'x' represents 10, because there's the 2 and the 5, that makes 7, plus 'x' which is 10, equals 17" and completed the equation by writing, $2x + 5 = 17, x = 10$.

Material use for solving the equation

Generative: The student placed two sticks on the table and said "That's 2x (made one group of five yellow discs and then a group of seventeen yellow discs beside the two sticks). That's 2x + 5 = 17 and you put negative five (added five green discs to the five yellow discs) so these are negative discs there and so that equals zero (removed the five yellow and five green discs). You put another negative five there (added five green discs to the seventeen discs) and you take five there (removed the five green discs and five of the seventeen yellow discs) and so that's twelve, $2x = 12$, but the 2 (removed one of the sticks and replaces it with two yellow discs) that's what it is, 2 times x. Put two there (added two yellow discs to the two yellow discs in front of the stick), they cancel down (removed both groups of two yellow discs), so you put two there (added two yellow discs to the group of twelve yellow discs), two goes into that (removed the two additional yellow discs), you halve it (removed six of the twelve yellow discs) and that equals x=6".

Illustrative: Fitting the answer - the student placed two empty cups on the table and said "You've got two cups with 'x' amount of discs inside (placed six discs in each of the two cups). We'll put 6 in there and then you add 5 outside (placed five green discs on the table outside the cups) and all the discs equal 17, so now you know that there's 6 discs inside the cup".

Following the mental - the student mentally calculated x to be six when presented with the equation, made a group of 17 discs when asked to use the materials, and said "These 17, take away 5 (removed 5 discs from the 17) and divide that by 2 (divided the remaining discs into two equal groups)".

Reversing - the student placed a group of 17 sticks on the table and said "So you'll have 17, I think - yes 17, then you take 5 away (removed 5 sticks) which is done in reverse and then you divide it by 2 (split remaining sticks into two equal groups) which leaves 6".